ISRAEL JOURNAL OF MATHEMATICS 130 (2002), 281–283

# SUBSPACES OF $l^\infty(\Gamma)$ WITHOUT QUASICOMPLEMENTS

BY

BENGT JOSEFSON

*Department of Mathematics, Linköpings University*
*S-581 83 Linköping, Sweden*
*e-mail: bejos@mai.liu.se*

ABSTRACT

J. Lindenstrauss proves in [L] that $c_0(\Gamma)$ is not quasicomplemented in $l^\infty(\Gamma)$ while H. P. Rosenthal in [R] proves that subspaces, whose dual balls are weak* sequentially compact and weak* separable, are quasicomplemented in $l^\infty(\Gamma)$. In this note it is proved that weak* separability of the dual is the precise condition determining whether a subspace, without isomorphic copies of $l_1$ and whose dual balls are weak* sequentially compact, is quasicomplemented or not in $l^\infty(\Gamma)$. Especially spaces isomorphic to $l_p(\Gamma)$, for $1 < p < \infty$, have no quasicomplements in $l^\infty(\Gamma)$ if $\Gamma$ is uncountable.

Let $X$ be a Banach space and let $Y$ be a closed subspace of $X$. A closed subspace $Z$ of $X$ is said to be a quasicomplement of $Y$ in $X$ if $Z \cap Y = \{0\}$ and $Z + Y$ is dense in $X$. This notation was introduced by F. Murray and Mackey [M] proved that in separable Banach spaces every closed subspace has a quasicomplement. The first example of a closed subspace without a quasicomplement was given by Lindenstrauss in [L], where he proves that $c_0(\Gamma)$ is not quasicomplemented in $l^\infty(\Gamma)$ if $\Gamma$ is uncountable. On the other hand Rosenthal proves in [R], Theorem 2.9, that closed subspaces with weak* separable and weak* sequentially compact dual balls (W*SCDB for short) always are quasicomplemented in $l^\infty(\Gamma)$. In this note we will prove that weak* separability of the dual is the property that determines the quasicomplementedness in $l^\infty(\Gamma)$ for a closed subspace with W*SCDB but without isomorphic copies of $l_1$. This shows that, besides $c_0(\Gamma)$, $l_p(\Gamma)$, for $p > 1$, are not quasicomplemented in $l^\infty(\Gamma)$ if $\Gamma$ is uncountable. Our proof is based on the fact that a set $D \subset l^\infty(\Gamma)$ is limited if $D$ does not

Received April 2, 2001

contain an isomorphic copy of the basis vectors of $l_1$; see [J1] or [J2] for a different proof. Recall that a set $D$ in a Banach space $X$ is called limited if any sequence of pointwise convergent (weak* convergent) continuous linear functionals on $E$ converges uniformly on $D$.

THEOREM 1: *Let $X$ be an injective Banach space and let $Y$ be a closed subspace whose dual balls are weak* sequentially compact. If $Y$ contains a closed subspace $Y_0$ without isomorphic copies of $l_1$ and whose dual is not weak* separable, then $Y$ has no quasicomplement in $X$. Especially spaces isomorphic to $c_0(\Gamma)$ or $l_p(\Gamma)$, $1 < p < \infty$, are not quasicomplemented in $l^\infty(\Gamma)$ if $\Gamma$ is uncountable.*

Before proving the theorem we recall some basic facts about limited sets. Obviously relatively compact sets are limited and limited sets are bounded. Banach spaces with no more limited sets than the relatively compact ones are called GP-spaces after Gelfand and Phillips. The following Lemma is well known:

LEMMA: *Banach spaces with W*SCDB are GP-spaces.*

*Proof:* Assume that $X$ is not a GP-space. Then there exists a limited set $D \subset X$ which is not relatively compact. That $D$ is not relatively compact ensures the existence of a bounded sequence $(\varphi_j)_{j\in\mathbf{N}} \subset X^\star$ separating a bounded sequence $(a_j)_{j\in\mathbf{N}} \subset D$, i.e., $\varphi_j(a_j) = 1$ but $\lim_{j\to\infty} \varphi_j(a_k) = 0$ for every $k$ (especially $(a_j)_{j\in\mathbf{N}}$ cannot be relatively compact). Since no subsequence of $(\varphi_j)_{j\in\mathbf{N}}$ converges uniformly on $D$ and $D$ is limited, $(\varphi_j)_{j\in\mathbf{N}}$ has no weak* converging subsequence. Thus $X$ has no W*SCDB and the Lemma is proved. ∎

We also need the following result which is an easy consequence of [J1] (note that limited sets are called weakly bounding in [J1]), where it is proved that a subset $D$ is limited in $l^\infty(A)$ if and only if $D$ contains no sequence isomorphic to the standard basis of $l_1$. Another proof is given in [J2].

THEOREM 2: *A subset $D$ of an injective Banach space $X$ is limited if and only if $D$ contains no sequence isomorphic to the standard basis of $l_1$.*

*Proof:* We may assume that $X \subset l^\infty(A)$ for some index set $A$. Let $i: X \to X$ be the identity mapping and $\hat{i}$ be its extension $\hat{i}: l^\infty(A) \to X$ which exists since $X$ is injective. If $D$ is not limited in $X$, $D$ cannot be limited in $l^\infty(A)$, because if $(\phi)_{j\in\mathbf{N}} \subset X^\star$ converges pointwise to 0 on $X$, then $(\phi \circ \hat{i})_{j\in\mathbf{N}} \subset (l^\infty(A))^\star$ converges pointwise to 0 on $l^\infty(A)$. Thus $D$ contains a sequence isomorphic to the standard basis of $l_1$ according to [J1] or [J2]. Conversely, if $D$ contains a sequence isomorphic to the standard basis of $l_1$, $D$ is not limited in $l^\infty(A)$ and hence not in $X$ by restriction. ∎

*Proof of Theorem 1:* Assume that there is a quasicomplement $Z$ to $Y$ in $X$ and let $T\colon X \to X/Z$ be the quotient map. Since $Y$ has a dense linear image in $X/Z$, $X/Z$ also has W*SCDB according to [D] p. 227. It then follows from the Lemma that $X/Z$ is a GP-space.

Let $B_{Y_0}$ denote the closed unit ball in $Y_0$. $B_{Y_0}$ is limited in $X$ according to Theorem 2 because $Y_0$ does not contain an isomorphic copy of $l_1$ by assumption.

The restriction map $T/Y_0$ is injective and hence its adjoint has a weak* dense range. Since the dual of $Y_0$ is not weak* separable, the adjoint cannot be a compact operator. Thus $T/Y_0$ is not compact either. Since $T(B_{Y_0})$ is not relatively compact it is not limited in the GP-space $X/\dot{Z}$.

Let $(\varphi_j)_{j\in\mathbf{N}} \subset (X/Z)^\star$ be a sequence of pointwise convergent continuous linear functionals which does not converge uniformly on $T(B_{Y_0})$. Then $(\varphi_j \circ T)_{j\in\mathbf{N}} \subset X^\star$ converges pointwise on $X$ but not uniformly on $B_{Y_0}$. Thus $B_{Y_0}$ is not limited in $X$ though $B_{Y_0}$ contains no sequence isomorphic to the standard basis of $l_1$. This contradicts Theorem 2 and completes the proof. ∎

*Remark:* Rosenthal's construction always gives $T(B_Y)$ relatively compact in $l^\infty(\Gamma)/Z$, $\Gamma$ being countable or not, if $Z$ is a quasicomplement of $Y$. The proof of Theorem 1 shows that we cannot do better if $Y$ does not contain an isomorphic copy of $l_1$.

## References

[D] J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics **92**, Springer, Berlin, 1984.

[J1] B. Josefson, *Bounding subsets of $l^\infty(A)$*, Journal de Mathématiques Pures et Appliquées **57** (1978), 397–421.

[J2] B. Josefson, *Ramsey classification of sequences in Banach spaces and applications*, preprint.

[L] J. Lindenstrauss, *On subspaces of Banach spaces without quasi-complements*, Israel Journal of Mathematics **6** (1969), 36–38.

[M] G. Mackey, *Note on a theorem of Murray*, Bulletin of the American Mathematical Society **52** (1946), 322–325.

[R] H. Rosenthal, *On quasicomplemented subspaces of Banach spaces, with an appendix on compactness of operators from $L_p(\mu)$ to $L_r(\nu)$*, Journal of Functional Analysis **4** (1969), 176–214.